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EXTENSIÓN DE LA INTEGRAL DIFUSA DE SUGENO POR MEDIO DE LÓGICA DIFUSA TIPO-2 GENERALIZADA

TESIS

Que para obtener el grado de Doctor en Ciencias

Presenta:

M.C. Gabriela Elizabeth Martínez Mendívil

Bajo la dirección de:
Dra. Elba Patricia Melin Olmeda

Co-dirigido por:
Dra. Olivia Mendoza Duarte

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EXTENSION OF THE FUZZY SUGENO INTEGRAL BY MEANS OF GENERALIZED TYPE-2 FUZZY LOGIC

THESIS

That to obtain the degree of Doctor in Sciences

Present:

M.Sc. Gabriela Elizabeth Martínez Mendívil

Under the direction of:

Dra. Elba Patricia Melin Olmeda

Co-directed by:

Dra. Olivia Mendoza Duarte

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This work is dedicated to

Ramón Antonio

Cynthia Paola and Daniel Eduardo

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Abstract

Aggregation operators enable to aggregate information from different sources of information to obtain a single representative output of the problem.

In this thesis, an extension of the aggregation operator of the generalized interval type-2 Sugeno integral is presented, and this method offers the best performance in applications of pattern recognition and edge detection. During the development of the extension of the operators of a generalized type-2 fuzzy system were applied to the fuzzy densities, the lambda calculation, fuzzy measurements and the fuzzy Sugeno integral.

The main goal of extending the Sugeno integral aggregation operator, is to give it the ability to handle higher levels of uncertainty by adding any member of sources and types of information in a wide variety of applications. In addition for demonstrating that the extended aggregation operator with generalized type-2 fuzzy logic presents a better performance than the traditional operator or extended operator with interval type-2 fuzzy logic. As part of the objectives, the proposed method was implemented in a modular neural network applied to face recognition and in an edge detector. Comparisons of the results obtained with respect to other aggregation operators were also made.

The extension of the Sugeno integral can be used in any application where it is necessary to add numerical information.

Keywords: Sugeno integral, fuzzy measures, fuzzy logic, generalized interval type-2 fuzzy logic, modular neural network, face recognition, edge detection.

Resúmen

Los operadores de agregación permiten agregar información procedente de diferentes fuentes de información para obtener una única salida representativa del problema.

En esta tesis se presenta una extensión al operador de agregación conocido como la Integral de Sugeno por medio de lógica difusa tipo-2 generalizada, este método ofrece un mejor rendimiento con aplicaciones de reconocimiento de patrones y detector de bordes. Durante el desarrollo de la extensión del operador de agregación, se aplicaron los operadores de un sistema difuso tipo-2 generalizado a las densidades difusas, el cálculo de lambda, medidas difusas y la integral difusa de Sugeno.

El principal objetivo de extender el operador de agregación de la integral de Sugeno, es el de proporcionarle la capacidad de manejar mayores niveles de incertidumbre al agregar cualquier cantidad de fuentes y tipos de información en una gran variedad de aplicaciones; además el demostrar que el operador de agregación extendido con lógica difusa tipo-2 generalizada presenta un mejor rendimiento que el operador tradicional o el extendido con lógica difusa tipo-2 por intervalos.

Como parte de los objetivos, el método propuesto fue implementado en una Red neuronal modular aplicado al reconocimiento de rostros y en un detector de bordes. También se realizaron comparaciones de los resultados obtenidos con respecto a otros operadores de agregación existentes.

La extensión de la integral de Sugeno puede ser utilizada en cualquier aplicación donde sea necesario agregar información numérica.

Palabras clave: Integral de Sugeno, lógica difusa, lógica difusa tipo-2 generalizada, red neuronal modular, reconocimiento de rostro, detección de bordes.

Universidad Autónoma de Baja California

FACULTAD DE CIENCIAS QUÍMICAS E INGENIERÍA

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C. Gabriela Elizabeth Martínez Mendivil
Pasante de: Doctor en Ciencias
Presente

El tema de trabajo y/o tesis para su examen profesional, en la Opción TESIS

Es propuesto, por los CC. Dras. Elba Patricia Melín Olmeda y Olivia Mendoza Duarte

Quienes serán los responsables de la calidad del trabajo que usted presente, referido al tema: EXTENSION DE LA INTEGRAL DIFUSA DE SUGENO POR MEDIO DE LÓGICA TIPO-2 GENERALIZADA

el cual deberá usted desarrollar, de acuerdo con el siguiente orden:

I.- INTRODUCCIÓN
II.- TEORÍA Y ANTECEDENTES
III.- MÉTODO PROPUESTO
IV.- SIMULACION DE RESULTADOS
V.- CONCLUSIONES Y TRABAJO FUTURO
VI.- REFERENCIAS

Dra. Elba Patricia Melín Olmeda
Directora de Tesis

Dr. José Luis González Vázquez
Secretario

Dra. Olivia Mendoza Duarte
Co-Directora de Tesis

Dr. Luis Enrique Palafox Maestre
Director

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1. Introduction

The process of information aggregation is a key element in any system in which it is necessary to have a decision making. Frequently, the result aggregated information considerably reduces the quantity of original information, however, perform it efficiently is one of the main tasks of various problems that handling large amounts of information, whose quality and accuracy can be very varied.

In previous work we consider operators that do not adequately reflect the process of aggregation of different sources or different criteria, so it is more convenient to use more robust aggregators, which are able to handle some degree of uncertainty, and some of them are capable of handling weights. The particular goal of aggregation operators is for combining information when they can be mathematically formalized. Aggregation operators are aimed at reducing a set of numbers into a unique representative value, as we can notice in Fig. 1.1. It is important to note that any aggregation or fusion process is based on numerical aggregation. An operator considers that the input variables are the information sources to combine and the output is the aggregation of the results.

In the bibliography we can find a variety of aggregation operators, the most traditional ones are: the geometric mean, arithmetic mean, weighted arithmetic mean, harmonic mean, however there are other more complex operators that used measures or weights, for instance the ordered weighted averaging (OWA) [1-2], weighted OWA (WOWA) [3], Choquet Integral [4] and Sugeno Integral [5].

The main contribution of this thesis is the proposed an extension of the Sugeno integral based on the operators of generalized type-2 fuzzy logic, by means of which one can achieve a better handling of the uncertainty than with the existing aggregation operators.

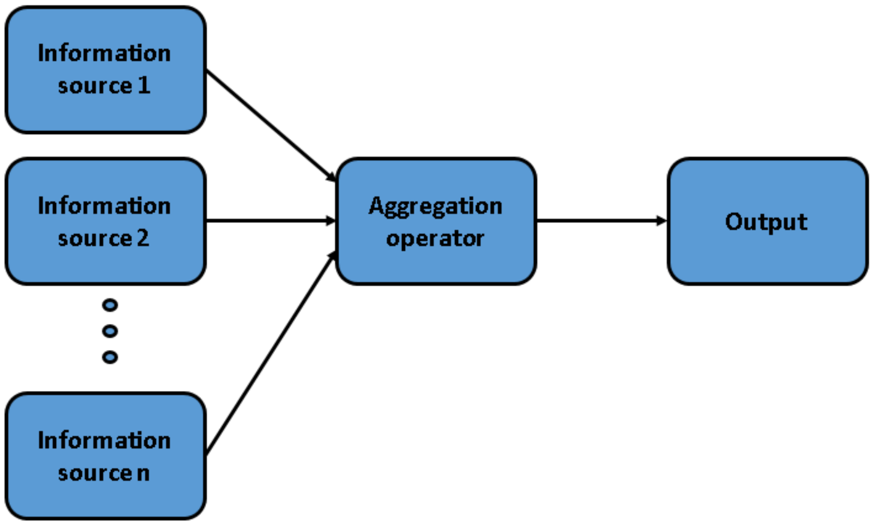


Fig. 1.1. Representation of the aggregation operator

The emergence of type-1 fuzzy systems [6-8] has allowed a great advance in various engineering applications, which has increased with the appearance of the interval type-2 fuzzy systems [9-11] and lately with the generalized type-2 fuzzy systems [12-16].The idea of using interval type-2 fuzzy systems and generalized type-2 fuzzy systems is that they can have a greater handling of uncertainty, so it is expected that better results will be obtained than with type-1 fuzzy systems. However, generalized fuzzy systems require a lot of computing power to carry out all their calculations. There are some methods that have been able to reduce the computing power of generalized systems, such as the α-planes [13-15] and zSlices [14] [16].

This thesis is organized as follows. In Section 2, the basic concepts of type-1 fuzzy logic, interval type-2 and generalized type-2 fuzzy logic are presented. Also we explained the Sugeno integral aggregation operator and the Interval type-2 Sugeno integral, which serve as a reference for developing the proposed method. Additionally we explains in detail the concepts of Modular neural networks (MNN) and edge detectors. The methodology used for developing the generalized type-2 Sugeno integral (GT2SI) is presented in Section 3. Section 4 offers presents simulation results with benchmark faces databases and benchmark images to illustrate the advantages of the proposed generalized type-2 Sugeno integral method and finally, Section 5 presents the conclusions and future work from the obtained results.

Next is a list of the publications generated from this thesis work:

1.1. Conference proceedings

- "Generalized Type-2 Fuzzy Logic in Response Integration of Modular Neural Networks". Gabriela E. Martínez, Olivia Mendoza, Juan R. Castro, Patricia Melin, Oscar Castillo. IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 2013.
- "Choquet Integral with Interval Type 2 Sugeno Measures as an Integration Method for Modular Neural Networks". Gabriela E. Martínez, Olivia Mendoza, Juan R. Castro, Patricia Melin, Oscar Castillo. World Conference on Soft Computing, (WConSC), 2014.
- "Response Integration in Modular Neural Networks using Choquet Integral with Interval Type-2 Sugeno Measures". Gabriela E. Martínez, Olivia Mendoza, Juan R. Castro. A. Rodríguez-Díaz, Patricia Melin, Oscar Castillo. Fuzzy Information Processing Society (NAFIPS) held jointly with 5th World Conference on Soft Computing (WConSC), 2015.
- "Comparison Between Choquet and Sugeno Integrals as Aggregation Operators for Modular Neural Networks". Gabriela E. Martínez, Olivia Mendoza, Patricia Melin, Fernando Gaxiola. IEEE World Congress on Computational Intelligence (IEEE WCCI). 2016.

- “Comparison Between Choquet and Sugeno Integrals as Aggregation Operators for pattern recognition”. Gabriela E. Martínez, Olivia Mendoza, Juan R. Castro. A. Rodríguez-Díaz, Patricia Melin, Oscar Castillo. 35th Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS), 2016.

1.2.Book chapters

- “Face Recognition with Choquet Integral in Modular Neural Networks”. Recent Advances on Hybrid Approaches for Designing Intelligent Systems, Studies in Computational Intelligence 547, Springer International Publishing Switzerland. pp 437-449. 2014.
- “Face Recognition with a Sobel Edge Detector and the Choquet Integral as Integration Method in a Modular Neural Networks”. Gabriela E. Martínez, Patricia Melin, Olivia D. Mendoza and Oscar Castillo. Design of Intelligent Systems Based on Fuzzy Logic, Neural Networks and Nature-Inspired Optimization. pp. 59-70. 2015.
- “Choquet integral and IT2 Choquet integral for edge detection”. Gabriela E. Martínez, Olivia Mendoza D., Juan R. Castro, Patricia Melin, Oscar Castillo. Nature Inspired Design of Hybrid Intelligent Systems, pp. 79-98. Abril 2016.
- “Choquet Integral with Interval Type-2 Sugeno Measures as an Integration Method for Modular Neural Networks”. Gabriela E. Martínez, Olivia Mendoza, Juan R. Castro, Patricia Melin, Oscar Castillo. Recent Developments and New Direction in Soft-Computing Foundations and Applications. Vol 342 of the series Studies in Fuzziness and Soft Computing. pp 71-86. 2016.

1.3. Journal articles

- "Extension of the Choquet Integral with Interval Type-2 Sugeno Measures applied in Modular Neural Networks". Gabriela E. Martínez, Olivia Mendoza, Juan R. Castro, Patricia Melin, Oscar Castillo. Applied Soft Computing. 2016.

2. Theory and Background

This work is based on the aggregation operator of the Sugeno integral which makes use of the Sugeno measures, which allows it to have an uncertainty management, however, the main contribution consists in the extension of the operator through the use of generalized type-2 fuzzy logic and α planes to add information with uncertainty; In this chapter we present some basic concepts about the work to understand better the general idea and the context.

2.1. Fuzzy logic

In a variety of real applications, when it is necessary to manipulate information, it is common to find problems due to the uncertainty which occurs when using inaccurate or imprecise data. Zadeh in 1965 propose the solution to the problem giving the definition of fuzzy set [17], and to complete the solution Michio Sugeno introduced the terms of fuzzy measure and fuzzy integral [5] as the most appropriate way to measure a certain degree of uncertainty, and these values that depend only on human subjectivity. The Fuzzy logic was defined by Zadeh in 1965.

2.1.1. Type-1 fuzzy logic

If X is a collection of objects denoted by x, then a "Fuzzy set" A in X is defined as a set of ordered pairs

$$\mathrm{A} = \{(\mathrm{x}, \mu_{\mathrm{A}}(x)) | \forall x \in X\} \tag{2.1}$$

where $\mu_{\mathrm{A}}(x)$ represent the membership function (MF) of the fuzzy set A.

2.1.2. Type-2 fuzzy logic

A Type-2 fuzzy set, denoted by $\widetilde{A}$, is characterized by a Type-2 membership function $\mu_{\tilde{A}}(x,u)$, where $x \in X$ and $u \in J_x \subseteq [0,1]$, and can be expressed as

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \tag{2.2}$$

in which $0 \leq \mu_{\tilde{A}}$ (x, u) ≤ 1. $\tilde{A}$ can also be expressed in an equivalent expression

$$\tilde{A} = \int_{x \in X} \int_{x \in j_x} \frac{\mu_{\tilde{A}}(x,u)}{x,u} J_x \subseteq [0,1] \tag{2.3}$$

where ∫∫ denotes the union over all admissible *x* and *u* [18].

The footprint of uncertainty (FOU) can be described as the upper and lower membership functions. An upper membership function and a lower membership function [18] are two Type-1 membership functions that bound the FOU of an interval Type-2 fuzzy set $\widetilde{A}$. The upper membership function is associated with the upper bound of the FOU ($\tilde{A}$), and is denoted by $\bar{\mu}_{\tilde{A}}(x)$, ∀x∈X.

$$\bar{\mu}_{\tilde{A}}(x) \equiv \overline{FOU(\tilde{A})} \qquad \forall x \in X \tag{2.4}$$

A lower membership function is associated with lower bound of the FOU ($\tilde{A}$), and is denoted by $\underline{\mu}_{\tilde{A}}(x)$, ∀ x∈X

$$\underline{\mu}_{\tilde{A}}(x) \equiv \underline{FOU(\tilde{A})} \qquad \forall x \in X \tag{2.5}$$

because the domain of a secondary membership function has been constrained in [0, 1], the lower and upper membership functions always exist [18].

The FOU ($\tilde{A}$) can also be expressed as:

$$FOU(\tilde{A}) = \bigcup_{\forall x \in X} \left[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)\right] \tag{2.6}$$

2.1.3. Generalized Type-2 fuzzy logic

Definition: a Generalized type-2 fuzzy sets denoted by $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u)$, where $x \in X$, $u \in J_x^u \subseteq [0,1]$ and $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$and can be represented by (2.7) [12-13] [19-21].

$$\tilde{A} = \left\{\left(x, \mu_{\tilde{A}}(x)\right) | x \in X\right\} = \left\{\left((x,u), \mu_{\tilde{A}}(x,u)\right) | \forall x \in X, \forall u \in J_x^u \subseteq [0,1]\right\} \tag{2.7}$$

if $\tilde{A}$ is continue is denoted by:

$$\tilde{A} = \left\{\int_{x \in X} \frac{\mu_{\tilde{A}}(x)}{x}\right\} \tag{2.8}$$

$$= \left\{\int_{x \in X} \int_{u \in J_x^u \subseteq [0,1]} \mu_{\tilde{A}}(x,u)/(x,u)\right\}$$

$$= \left\{\int_{x \in X} \left[\int_{u \in J_x^u \subseteq [0,1]} f_x(u)/u\right]/x\right\}$$

where ∬ represent the union for x and u. x is primary domain, J_x is the secondary domain, $\mu_{\tilde{A}}(x)$ is the secondary membership function at x is called a vertical slice of $\tilde{A}$[19] and all secondary grades are represented by $\mu_{\tilde{A}}(x,u) \in [0,1]$; and if $\tilde{A}$ is discreet is denoted by:

$$\widetilde{A} = \left\{\sum\nolimits_{x \in X} \boldsymbol{\mu}_{\tilde{A}}(\boldsymbol{x})/\boldsymbol{x}\right\} \tag{2.10}$$

$$= \left\{\sum_{x \in X} \sum_{u \in J_x^u \subseteq [0,1]} \boldsymbol{\mu}_{\tilde{A}}(\boldsymbol{x},\boldsymbol{u})/(\boldsymbol{x},\boldsymbol{u})\right\}$$

$$= \left\{ \sum_{x \in X} \left[\sum_{u \in J_x^u \subseteq [0,1]} f_x(u)/u \right] / x \right\}$$

$$= \left\{ \sum_{i=1}^{N} \left[\sum_{k=1}^{M_i} f_{x_i}(u_{ik})/u_{ik} \right] / x_i \right\}$$

where ΣΣ denote the union of x and u.

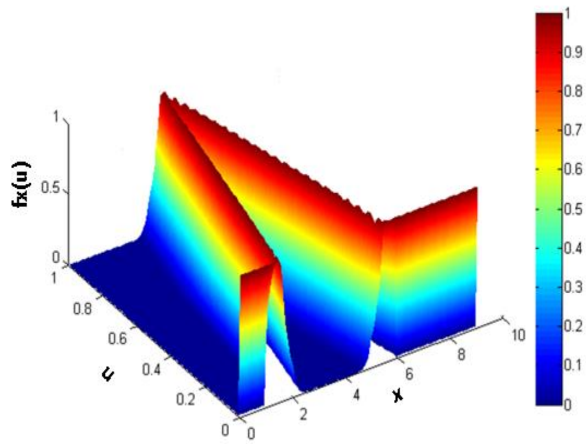


Fig. 2.1. Generalized type-2 membership function

We can visualize the representation of a generalized type-2 membership function in Fig. 2.2 that is associated with the third dimension; the footprint of uncertainty (FOU) of the function can be seen in the Fig.2.3.

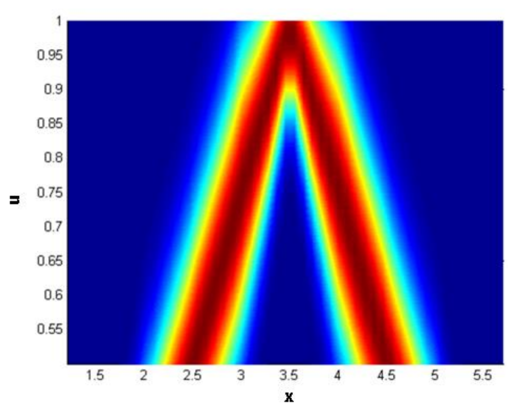


Fig. 2.2. FOU of a general type-2 membership function

The generalized type-2 fuzzy systems have been successfully applied in [22-25].

2.1.3.1. Alpha Planes

An α plane for a generalized type-2 fuzzy set Ã is denoted by $\tilde{A}_\alpha$ and can be defined as the union of all primary membership functions of Ã, where the secondary membership degrees are greater or equal to α (0 ≤ α ≤ 1) and can be represented by (2.16) [12-13], and can be graphically illustrated in Fig. 2.4.

$$\tilde{A}_\alpha = \{(x,u), \mu_{\tilde{A}}(x,u) \geq \alpha | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \tag{2.16}$$

$$= \int_{\forall x \in X} \int_{\forall u \in J_x} \{(x,u) | fx(u) \geq \alpha\}$$

where the union of all the α alpha planes is represented by (2.A), and $R_{\tilde{A}_\alpha}$ represent a horizontal slice. In Fig. 2.3 we can appreciate α alpha planes made to generalized membership functions.

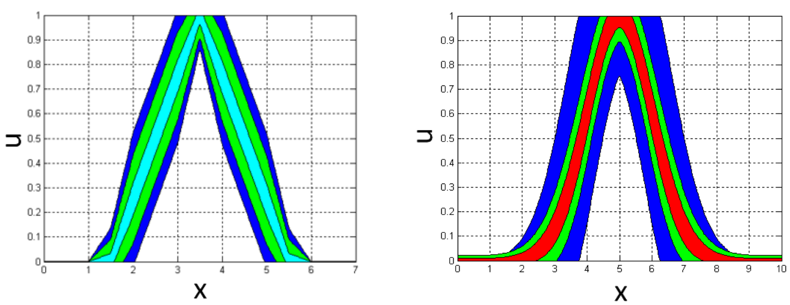
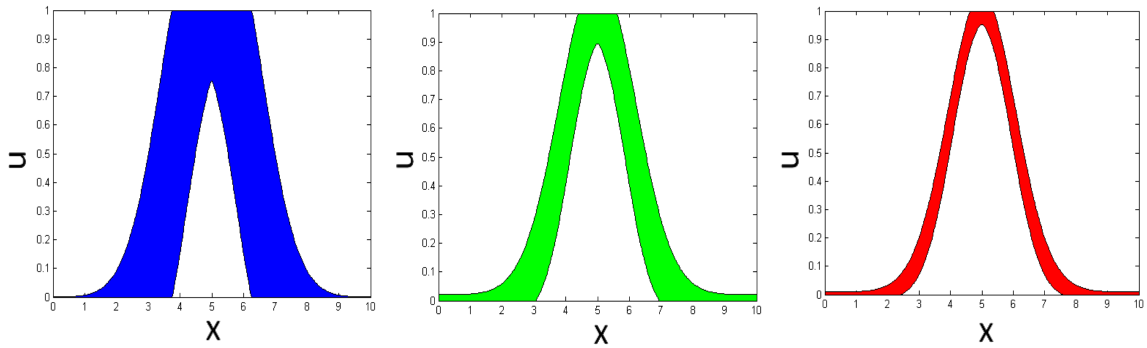


Fig. 2.3. α plane in generalized type-2 membership functions

Also in Fig. 2.4 we can appreciate the representation of three α plane in different points, the first α plane was realized in the point 0.17 (blue), the second α plane was in 0.46 (green) and the third in the point 0.99 (red).

$$\tilde{A} = \bigcup_{\alpha \epsilon [0,1]} R_{\tilde{A}_\alpha} \tag{2.A}$$



a)α = 0.17 b) α = 0.46 c) α = 0.99

Fig. 2.4. Cuts at certain points of the GMF

2.1.3.2. Generalized type-2 fuzzy systems based on α-planes

Due to the computational complexity of a generalized diffuse type-2 system, the calculation is carried out by means of an approximation using α planes or zSlices. The main idea of this thesis is based on the operators of a diffuse generalized type-2 system based on α planes.

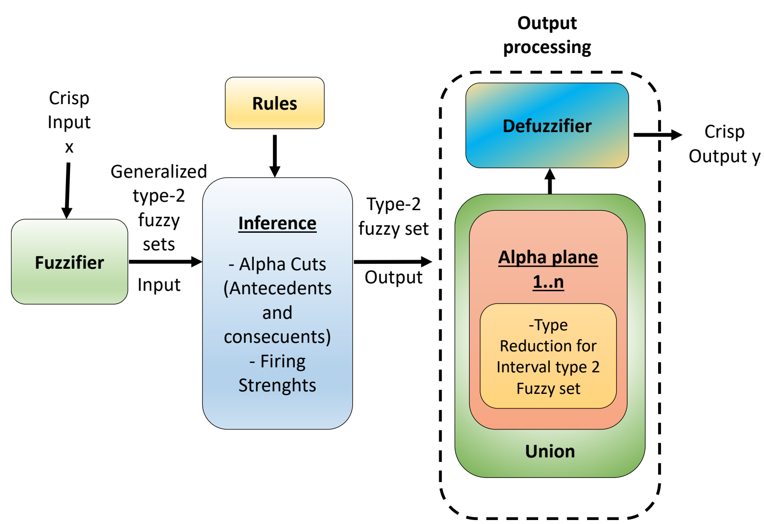


Fig 2.5 Diagram of Generalized type-2 fuzzy logic system based in α planes

In Fig. 2.5 is presented a diagram where we can visualize the generalized type-2 fuzzy system, here we can appreciate its main elements, which are: the fuzzifier process, fuzzy rules, inference, type reducer, and the process of defuzzification; this representation is based in the approximation of α planes.

2.1.3.3. Parametrization of a Generalized Type-2 Membership Function (GT2MF)

In this section we defined a generalized triangular membership function in the primary with a Gaussian function in the secondary. The parameters of the membership functions were obtained by (2.11) and we defined the generalized membership function as follows: $\overline{\mu}(x,u) = trigausstype2(x,u,[a_1,b_1,c_1,a_2,b_2,c_2,\rho])$

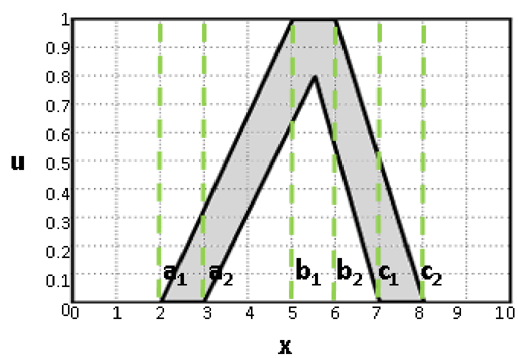


Fig. 2.6. Interval type-2 membership function

where *x* is the partition of the primary membership function and *u* represent the domain of the secondary membership function,

$$\tilde{\mu}(x,u) = exp\left[-\frac{1}{2}\left(\frac{u-px}{\sigma_u}\right)^2\right] \tag{2.11}$$

in which

$$\sigma_u = \frac{1+\rho}{2\sqrt{3}}\delta + \varepsilon \tag{2.12}$$

where ρ, is fraction uncertain of the support secondary membership function

$$\delta = \overline{\mu}(x) - \underline{\mu}(x) \tag{2.13}$$

$$px = max\left(min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right) \tag{2.14}$$

$$a = \frac{a_1 + a_2}{2}, \quad b = \frac{b_1 + b_2}{2}, \quad c = \frac{c_1 + c_2}{2} \tag{2.15}$$

where “trigausstype2” stands for the Gaussian interval type-2 membership function with uncertain mean. This generalized membership function was the one used for the development of the proposed method.

2.1.3.4. Meet and Join generalized

$$\boldsymbol{\mu}_{\tilde{\tilde{A}}}(\boldsymbol{x}) \sqcap \boldsymbol{\mu}_{\tilde{\tilde{B}}}(\boldsymbol{x}) = \left\{\left[\int_{\boldsymbol{u}\in \boldsymbol{J}_{\boldsymbol{x}}^{\boldsymbol{u}}} \int_{\boldsymbol{w}\in \boldsymbol{J}_{\boldsymbol{x}}^{\boldsymbol{w}}} \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{u}) \tilde{*} \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{w})/(\boldsymbol{u} \wedge \boldsymbol{w})\right]\right\} \tag{2.17}$$

$$\boldsymbol{\mu}_{\tilde{\tilde{A}}}(\boldsymbol{x}) \sqcup \boldsymbol{\mu}_{\tilde{\tilde{B}}}(\boldsymbol{x}) = \left\{\left[\int_{\boldsymbol{u}\in \boldsymbol{J}_{\boldsymbol{x}}^{\boldsymbol{u}}} \int_{\boldsymbol{w}\in \boldsymbol{J}_{\boldsymbol{x}}^{\boldsymbol{w}}} \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{u}) \tilde{*} \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{w})/(\boldsymbol{u} \vee \boldsymbol{w})\right]\right\} \tag{2.18}$$

2.2. Fuzzy Measures and the Sugeno Integral

As far as aggregation operators are concerned, in the last few years, diffuse integrals and diffuse measures have had much boom in the research area that is why it is of great interest to work with this type of operators which handle measurements.

2.2.1. Sugeno Measures

A special type of monotonic measures are the Sugeno λ-measures [26-27] defined as follows: If we have a finite set x={x_1, x_2, …, x_n}, a fuzzy measure μ with respect to the dataset X, can be defined as a function $\mu: 2^x \to [0,1]$ that must fulfill the following conditions:

1) μ(X) = 1; μ(∅) = 0 (Boundary conditions)

2) If A ⊆ B, then μ(A) ≤ μ(B) (Monotonicity)

where A and B are subsets of X.

A Sugeno measure is a fuzzy measure or λ-fuzzy, if it satisfies the condition (1) of addition for some λ>-1.

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\, \mu(A)\mu(B) \quad (2.19)$$

Equation (2.19) is usually called the λ-rule. When X is a finite set and the values μ ({x}), called fuzzy densities, are given for each x ∈ X, these densities are interpreted as the importance or relevance of the individual information sources. The measure of a set A of the information sources is interpreted as the importance of that subset of sources towards answering or solving a particular question or problem [28].

The value of μ(A) for each A ⊂ P(X), can be determined by the recurrent application of the λ-rule. This value can be expressed in the following way

$$\mu(\mathrm{A}) = \left[\prod_{x \in A}(1 + \lambda\mu(\{\mathrm{x}\})\right]/\lambda \tag{2.20}$$

We can notice that once the values of the fuzzy densities μ({x}) are assigned for each x ∈ X, the value of λ can be calculated by using the constraint μ({x}) = 1. So that by applying this restriction (2.20) we obtain (2.21)

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda\mu(\{x_i\})) \tag{2.21}$$

Once the densities are known, the λ parameter can be computed using (2.21) and is specific to this class of measures.

Sugeno proved that this polynomial has a real root greater than −1 and several researchers have observed that this polynomial equation is easily solved numerically. By property (2.19), specifying the *n* different densities, thereby reducing the number of free parameters from $2^n - 2$ to *n* [26], the value of parameter λ is determined with the help of the following theorem [29]:

Theorem 1: Let μ({x}) < 1 for each x ∈ X and let μ({x}) > 0 for at least two elements of X. Then (2.21) determines a unique parameter λ in the following way.

- If $\sum_{x \in X} \mu(\{x\}) < 1$, then λ is equal to a unique root of the equation in the interval (0,∞).
- If $\sum_{x \in X} \mu(\{x\}) = 1$, then λ = 0; that is the unique root of the equation.
- If $\sum_{x \in X} \mu(\{x\}) > 1$, then λ is equal to a unique root of the equation in the interval (−1, 0).

When μ is a λ-fuzzy measure, the values of $\mu(A_i)$ can be computed by means of (2.20), or recursively, after a descendent reordering of the sets X and μ({x}), with respect to the values of the elements of set X [30].

There are two types of Integral that can perform the calculation of Sugeno measures, these are the Sugeno Integral (SI) and the Choquet Integral, which have already been used in a variety of applications [31].

2.2.2. Sugeno Integral

Using the concept of fuzzy measures, Sugeno also proposed the concept of a fuzzy Integral as nonlinear functions defined with respect to fuzzy measures such as the λ-fuzzy measure. The Sugeno Integral (SI) generalizes the "max-min" operators. One can interpret the fuzzy Integral as finding the maximum degree of similarity between the target and the expected value as shown in (2.22).

$$Sugeno_{\mu}(x_1, x_2, \dots, x_n) = \max_{i=1..n}\left(\min\left(D(x_{\sigma(i)}), \mu(A_{\sigma(i)})\right)\right) \tag{2.22}$$

where $x_{\sigma(i)}$ indicates that the indices of the data $D(x_{\sigma(\mathrm{i})})$ have been permuted as $0 \leq D(x_{\sigma(1)}) \leq D(x_{\sigma(2)}) \leq \cdots \leq D(x_{\sigma(n)}) \leq 1$, and where $A_{\sigma(i)} = \{A_{\sigma(1)}, \dots, A_{\sigma(n)}\}$.

The Sugeno Integral can be used to solve problems that consider a finite set of *n* elements $X=\{x_1, \dots, x_n\}$.

2.2.3. Fuzzy measures and Sugeno Integral for interval Type-2 fuzzy sets

In this section is defined the interval type-2 Sugeno integral (IT2SI).

2.2.3.1. Fuzzy measures with interval Type-2 fuzzy sets

To calculate the diffuse density of each information source, it takes the maximum value of each X_i and is added a footprint of uncertainty (FOU) to form a Type-2 fuzzy set. We need to add a FOU to create a density based on fuzzy interval. Equation (2.24) can be used to approximate the center of the interval and equations (2.25) and (2.26) for calculate the values of the upper and lower interval for each fuzzy density. Note that the domains for $\mu_U(x_i)$ and $\mu_L(x_i)$ are given in Theorem 1 [32].

The calculation of the left and right fuzzy densities can be defined as follows:

$$\mu_c(x_i) = \max(X_i) \tag{2.24}$$

$$\mu_U(x_i) = \begin{cases} \mu_c(x_i) - FOU_\mu/2; \; if \, \mu_c(x_i) > FOU_\mu/2 \\ \varepsilon_U \quad \text{otherwise} \end{cases} \tag{2.25}$$

$$\mu_L(x_i) = \begin{cases} \mu_c(x_i) + FOU_\mu/2; \; if \, \mu_c(x_i) < (1 - FOU_\mu/2) \\ \varepsilon_L \quad \text{otherwise} \end{cases} \tag{2.26}$$

where i = 2, 3, . . . , *n*, and *n* represents the number of information sources, μ_c is the central fuzzy density, μ_U is the upper fuzzy density, μ_L is the lower fuzzy density, and FOUμ is the footprint of uncertainty added to the fuzzy densities. The ε_U and ε_L values are determined by the following conditions, in order to satisfy Theorem 1:

- ε_U is the smallest number in (0, $\mu_c(x_i)$) depending on the application.
- ε_L is the biggest number in (($\mu_c(x_i)$, 1) depending on the application.

The λ_U upper and λ_L lower parameters for each side of the interval can be calculated with (2.27) and (2.28)

$$\lambda_U + 1 = \prod_{i=1}^{n} (1 + \lambda_U \mu_U (\{x_i\})) \tag{2.27}$$

$$\lambda_L + 1 = \prod_{i=1}^{n} (1 + \lambda_L \mu_L (\{x_i\})) \tag{2.28}$$

Once the λ_U and λ_L are obtained the upper $\mu_U(A_i)$ (2.29) (2.30) and the lower $\mu_L(A_i)$ (2.31)(2.32) fuzzy measures can be calculated by extending the recursive formula as follows:

$$\mu_U(A_1) = \mu_U(x_1) \tag{2.29}$$

$$\mu_U(A_i) = \mu_U(x_i) + \mu_U(A_{i-1}) + \lambda_U \mu_U(x_i)\mu_U(A_{i-1}) \tag{2.30}$$

$$\mu_L(A_1) = \mu_L(x_1) \tag{2.31}$$

$$\mu_L(A_i) = \mu_L(x_i) + \mu_L(A_{i-1}) + \lambda_L \mu_L(x_i)\mu_L(A_{i-1}) \tag{2.32}$$

2.2.3.2. Sugeno Integral with interval Type-2 fuzzy sets

If we extend Equation (2.22) using intervals we obtain the following expression:

$$Sugeno_U = \max_{i=1..n}\left(\min\left(D_U(x_i), \mu_U(A_i)\right)\right) \tag{2.33}$$

$$Sugeno_L = \max_{i=1..n}\left(\min\left(D_L(x_i), \mu_L(A_i)\right)\right) \tag{2.34}$$

where $Sugeno_U$ and $Sugeno_L$ represent the upper and lower interval extremes of the Sugeno integral.

Once the upper and lower intervals are calculated, the next step is to calculate the average of both values to obtain the IT2SI using (2.35)

$$Sugeno = (Sugeno_U + Sugeno_L)/2 \tag{2.35}$$

2.3. Edge detection

One of the main steps that is performed in many applications of digital images processing is the extraction of patterns or significant elements. One of the main features that is usually obtained is the edge or outline of an object of the image, which provides information of great importance for later stages.

Edge detection is a process in digital image analysis that detects changes in light intensity, and it is an essential part of many computer vision systems. The edge detection process is useful for simplifying the analysis of images by dramatically reducing the amount of data to be processed [32]. An edge may be the result of changes in light absorption (shade/color/texture, etc.) and can delineate the boundary between two different regions in an image.

Table 2.1. Edge detection methods

Traditional methods	**Computational intelligence techniques**
Morphological gradient	Morphological gradient with fuzzy system
Sobel	Morphological gradient with interval type-2 fuzzy system
Prewitt	
Roberts	Sobel with fuzzy system
Laplacian	
Canny	Sobel with interval type-2 fuzzy system
Kirsch	
LoG	Sugeno integral
Zero crossing	Interval type-2 Sugeno Integral

The resultant images of edge detectors preserve more details of the original images, which is a desirable feature for a pattern recognition system. In the literature there are various edge detectors, amongst them, the best known operators are the Morphological gradient, Sobel [33], Prewitt [34], Robert [35], Canny [32] and Kirsch [36]. There are also edge detection methods that combine traditional detection methods with different techniques or even intelligent methods, which have been successful such as type-1 fuzzy systems [37], interval type-2 fuzzy systems combined with the Sobel operator [38], interval type-2 fuzzy systems and the morphological gradient [39-40], Sugeno integral and interval type-2 Sugeno integral[40], among others. In Table 1, a summary of existing edge detection methods is presented.

The edges of a digital image can be defined as transitions between two regions of significantly different gray levels, and they provide valuable information about the boundaries of objects and can be used to recognize patterns and objects, segmenting an image, etc.

2.3.1. Edge detector based on Gradient

Gradient operators are based on the idea of using the first or second derivative of the gray level. Based on an image *f(x, y)*, the gradient of point *(x, y)* is defined as a gradient vector (∇f) and is calculated as follows:

$$\nabla f = \begin{bmatrix} Gx \\ Gy \end{bmatrix}$$
$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \quad (2.36)$$

where, the gradient magnitude vector (*mag(*∇f*)*) is calculated with:

$$mag(\nabla f) = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{1/2} \quad (2.37)$$
$$mag(\nabla f) = \left[G_x^2 + G_y^2\right]^{1/2}$$

In this case, we are going to use G_i instead of $mag(\nabla f)$, in other words we apply (2.37) for a matrix of 3x3 as it is shown in Fig. 2.7. The values for z_i are obtained using (2.38), and the possible direction of edge G_i with (2.39 - 2.42). The G gradient can be calculated using (2.43).

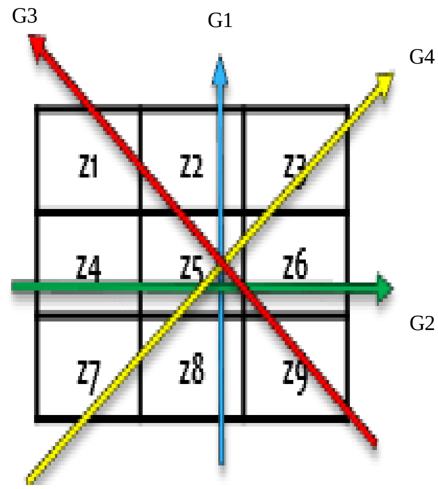


Fig. 2.7. Matrix of 3x3 of the index Z_i, that is indicating the calculation of the gradient in the four directions.

$$z_1 = (x-1, y-1) \qquad z_6 = (x, y+1)$$
$$z_2 = (x-1, y) \qquad z_7 = (x+1, y-1)$$
$$z_3 = (x-1, y+1) \qquad z_8 = (x+1, y) \tag{2.38}$$
$$z_4 = (x, y-1) \qquad z_9 = (x+1, y+1)$$
$$z_5 = (x, y)$$

$$G1 = \sqrt{(z5-z2)^2 + (z5-z8)^2} \tag{2.39}$$

$$G2 = \sqrt{(z5-z4)^2 + (z5-z6)^2} \tag{2.40}$$

$$G3 = \sqrt{(z5-z1)^2 + (z5-z9)^2} \tag{2.41}$$

$$G4 = \sqrt{(z5-z7)^2 + (z5-z3)^2} \tag{2.42}$$

$$G = G1 + G2 + G3 + G4 \tag{2.43}$$

2.3.1.1.FOM metrics

Once edge detection with a particular technique is performed, it is necessary to use some evaluation method to determine whether the result is good or better than other existing edge detection methods in digital images.

In the literature, we can find different metrics to evaluate the detected edges, to perform these evaluations, a measure of differences that is frequently used is the quadratic average distance proposed by Pratt [41], and it is also called Figure of Merit (FOM). This measure represents the variation that exists from a real point (calculated) from the ideal edge.

$$FOM = \frac{1}{\max(I_I, I_A)} \sum_{i=1}^{I_A} \frac{1}{1 + \propto d_i^2} \qquad (2.44)$$

where I_A represents the number of detected edge points, I_I is the number of points on the ideal edge, d(i) is the distance between the current pixel and its correct position in the reference image and α is a scaling parameter (normally 1/9)**.** To apply (2.44) we require the synthetic image and its reference. A FOM=1 corresponds to a perfect match between the edge ideal and the detected edge points.

2.4. Modular neural network

The focus is on the aggregation operators that are based on measures, in particular the GIT2SI that is applied to a modular neural network (MNN) for the case of face recognition. MNNs use others techniques to add the information of the modules, like type-1 and type-2 fuzzy systems [42-44], the fuzzy choquet integral [45], the Sugeno integral[46] a probabilistic sum integrator [47], a novel Bayesian learning method [48] and self-organizing maps [49].

Table 2.2.Combining information sources

Integration methods	Aggregation operators
The winner takes it all	Arithmetic mean
Linear combination of results	Geometric mean
Voting mechanisms	Weighted mean
Models in series	OWA
Discrete logic	OWA weighted
Type-1 and interval Type-2 fuzzy logic	Harmonic mean
Fuzzy Sugeno integral	Choquet integral
Interval Type-2 Fuzzy integral	Sugeno integral
Self-organizing maps	Interval type-2 Sugeno integral
	Bayesian learning method
	Probabilistic sum integrator

2.4.1. Methodology for face recognition

The Sugeno integral has been used in a variety of applications; in this case we use the GT2SI as an aggregation operator for a MNN applied to face recognition. The experiment was performed as follows: to each of the selected databases we applied a preprocessing step by using edge detectors to highlight the characteristics of the information. Subsequently, we performed the image division into 3 segments for training with each of the modules of the MNN. Then the cross-validation method was applied for the selection of the training and testing data, finally we performed the aggregation process through the implementation of the GT2SI for achieving the recognition. This procedure can be found in Fig. 2.8.

In this case, we considered face databases for the experiment; however, the proposed aggregation operator can be used in any application where it is necessary to perform the process of numerical aggregation.

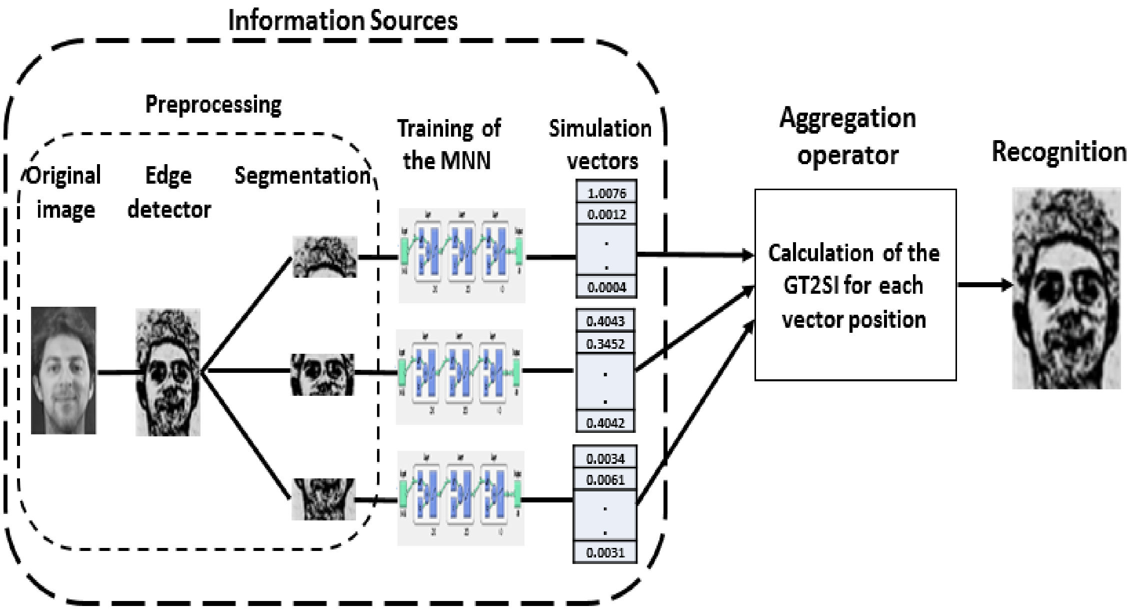


Fig. 2.8. Proposed architecture for face recognition.

The sources of information that should be aggregated are represented by the outputs of each of the MNN modules. Of course, it is necessary to assign a fuzzy density $\mu(\{x_i\})$ to each of the sources of information.

The calculation of fuzzy densities was performed using (2.45) where *i, j* and *k* represent the values of the fuzzy densities assigned to each information source. For the experiments, the *i*, *j* and *k* variables were increased by 0.4 in each iteration.

$$\overrightarrow{FD} = \sum_{i=0.1}^{0.9} \sum_{j=0.1}^{0.9} \sum_{k=0.1}^{0.9} [i, j, k] \qquad (2.45)$$

In the experiments we performed 27 tests per each simulation with the edge detectors. Each test is the result of the permutations made on *i*, *j* and *k* with the values of 0.1, 0.5 and 0.9, which represent the fuzzy densities assigned to each information source, in this case for each module. The parameters of the fuzzy densities are in this case assigned arbitrarily, however the selected parameters are not optimized, so it is necessary to apply some methodology to find the optimal densities for the particular problem.

2.4.2. Face Databases

To validate the proposed approach, we used two face databases, the first one is the ORL database [50]. This database has faces of 40 people with 10 samples of each individual. All the images of the ORL face database are in a frontal position with a rotation of up to about 20 degrees, for some a tolerance of tilting and against a dark homogeneous background. The images of some subjects were taken at different times and have varying lighting, facial expressions with open and closed eyes, smiling or not smiling and facial details, such as glasses or no glasses. To each of the faces of the database we apply a pre-processing step by using the Morphological Gradient edge detector [51] in order to highlight the main features. The images of the ORL database have a dimension of 112 x 92. The Cropped Yale database has faces of 38 people with 10 samples of each person, each image has a size of 168 x 192, and a similar process is applied.

2.4.3. Pre-processing with the Morphological Gradient edge detector

When working with a pattern recognition system is necessary keep as much information as possible, so that making use of an edge detector method we have that the resulting images retain more detail than the original images. We can define an edge detection method as a procedure that consists on identifying variations that exists in the light intensity, which can be used as a factor to determine properties or characteristics of the elements present in the faces.

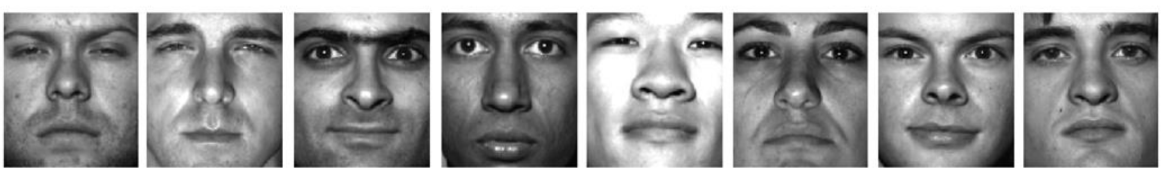
a)

b)

c)

Fig. 2.9. a) Original faces of the ORL database, b) image after applying MGT1FLS, c) Image after applying MGIT2FLS.

Different researchers have dealt with the problem developing edge detectors, in [51-52] has been shown that performing a pre-processing to the images, using an edge detector, and significantly improves the recognition rate. For this reason, in this case the Type-1 and interval Type-2 Morphological Gradient edge detectors are used for each one of the face databases. In Fig. 2.10 and Fig. 2.11, in a) we can appreciate the original images, in b) we can find the images after applying the Type-1 Morphological Gradient and in c) we can observe the images after applying the interval Type-2 Morphological Gradient.



a)

b)

c)

Fig. 2.10. a) Original faces of the Cropped Yale database, b) images after applying the MGT1FLS method, c) Images after applying the MGIT2FLS method.

2.4.4. Cross validation method

In the realization of the experiments we established the quantity of persons as *p* and *s* as the number of samples per person. For each of the databases, tests were made using the k-fold cross validation method, with *k*=5.

We can generalize the calculation of the fold size *m* or number of samples in each fold, dividing the total number of samples *s* for each person between the folds number, and then multiplying the result by the people number *p* (2.46). The training data set size *i* (2.47) can be calculated as the number of samples in *k-1* folds m, and the test data set size *t* (2.48) is the number of samples in only one fold [53].

$$m=(s/k)*p \tag{2.46}$$

$$i=m*(k-1) \tag{2.47}$$

$$t=m \tag{2.48}$$

The total quantity of samples used for each person has value of ten for both the ORL and the Cropped Yale databases; then if the size *m* of each five-fold is two, the quantity of samples for the training of each people is eight and for testing is two.

Table 2.3. Distribution of the data in the folds

Database	People quantity (p)	Samples per person (s)	Fold size (m)
ORL	40	10	80
Cropped Yale	38	10	76

The total number of samples used for each person are of eight for the Cropped Yale databases; then if the size m of each five-fold is two, the number of samples for the training for each person is eight and for testing two, as can be noted in Table 2.3.

2.4.5. Training of the MNN

The concept of artificial neural networks (ANN) was introduced by W. S. McCullough and W. Pitts in 1943 [54] inspired by the biological neural networks of the human brain.

An ANN is defined as an information processing system that has some performance characteristics similar to the biological neurons. The ANNs are considered mathematical models of the neural biology or the human cognition [55]. We applied the neural network concept due to the ability to learn from data, yet, there are engineering problems or applications that cannot be computed on a single neural network due of its difficulty or the amount information that is handled. In these cases we can divide the problem into sub-problems or subtasks that are less complex and use a MNN so that each module is able to solve a part of the problem, thus reducing the corresponding complexity [57-59]. So it is necessary to apply an aggregation operator that allows to integrate the information from the different sources (modules of the MNN) to give the total solution to the problem. We trained a modular neural network of 3 modules with each face database. To each of the images the MGT1FLS and MGIT2FLS edge detectors are applied, then each image is split into three horizontal sections. Each section of the image is used as training data for the MNN, and after that we proceed to perform the integration of information using (23) (24) and (27). This methodology is described in more detail in Fig. 2.12.

In Table 2.4 we can find the training parameters assigned to each monolithic neural network of the MNN, which were proposed in [60], and in Fig.2.13. we can appreciate the architecture of a monolithic neural network of the MNN.

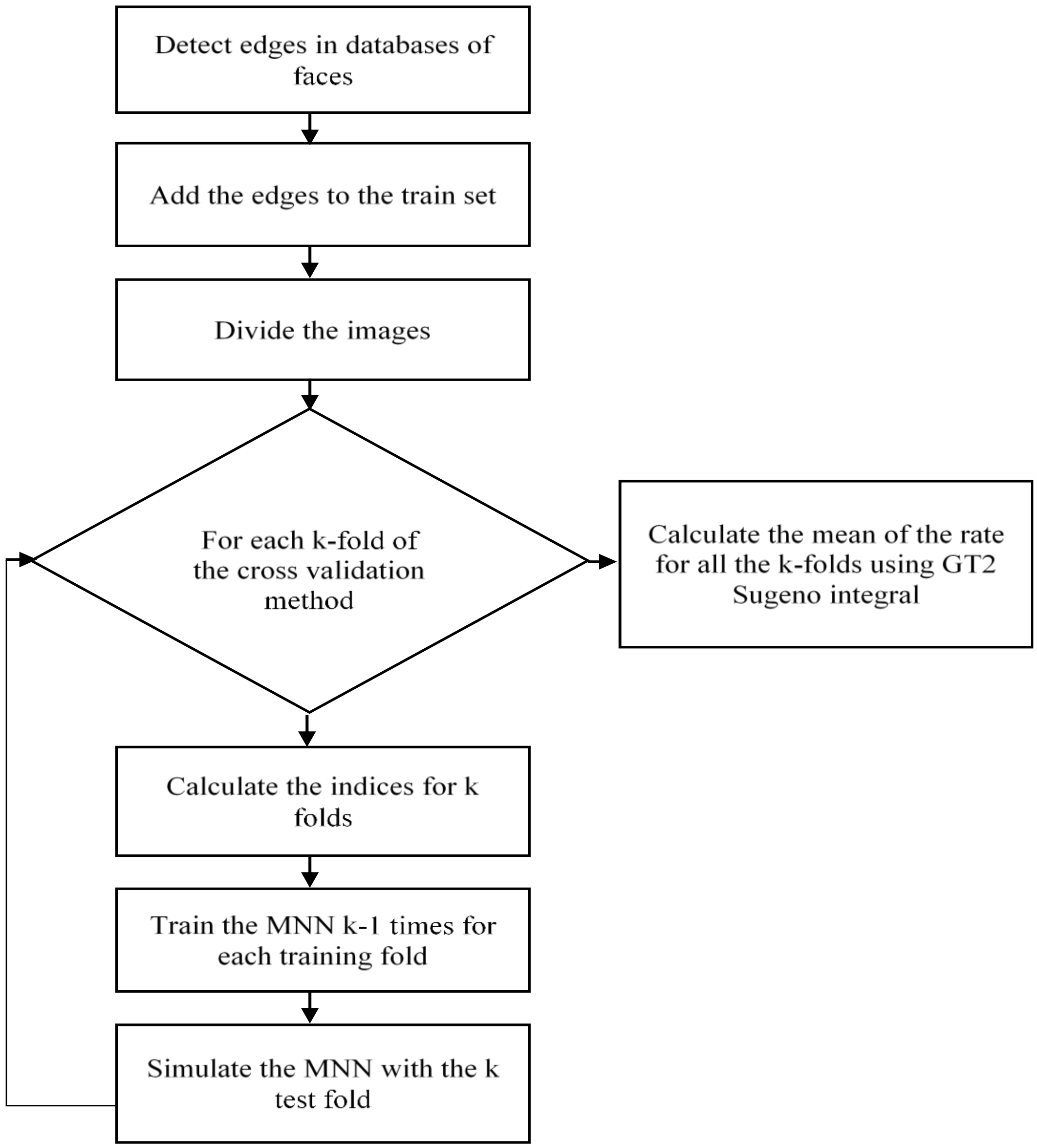


Fig. 2.11. Procedure for recognition

Table 2.4. Parameters of the MNN.

Architecture	Parameters
Training method	Traingdx
Hidden layers	2
Neurons per layer	200
Epochs	500
Error goal	0.0001

The selection of training data was performed using the cross-validation method, which goal is to ensure that the results obtained are independent of the partition used to select both the training and the test data.

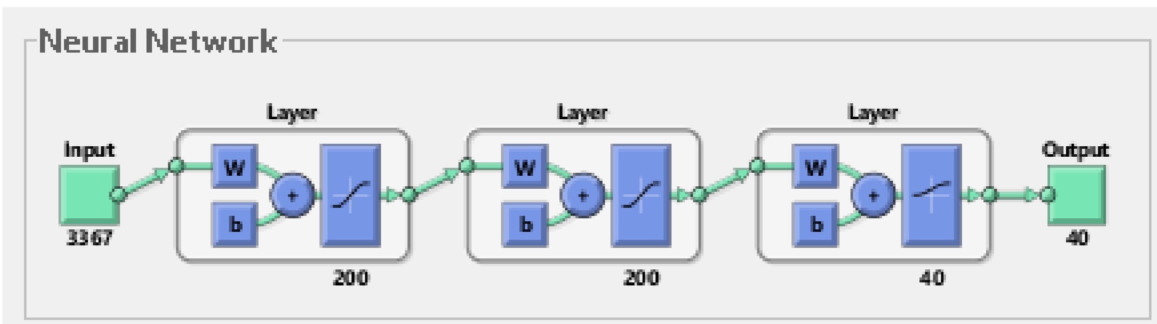


Fig. 2.12. Architecture of each monolithic neural network

2.4.6. Experiment

The work consists on obtaining a data set of the ORL and Cropped Yale benchmark face databases by applying each of the edge detectors, and then train a MNN to compare the percentages of recognition applying the k-fold cross validation method [61]. The sequence of steps performed can be found in the diagram of Fig. 2.12.

3. Proposed method

In this chapter, the proposed method for extending the Sugeno integral with the operators of the Generalized type-2 fuzzy system is explained and illustrated.

3.1. Generalized interval type-2 Sugeno integral

The main objective of this work is to extend to the Sugeno integral using the generalized type-2 fuzzy systems. The series of steps used to perform the extension are presented below:

1) First, depending on the problem, we need to define the number of information sources n, information sources $D(x_i)$, and the fuzzy densities assigned to each information source M(xi) ∈ (0,1).

2) To obtain a GT2SI it is necessary to evaluate each $D(x_i)$ and $M(x_i)$ with a generalized type-2 membership function with a triangular in the primary and a Gaussian in the secondary, the function $\tilde{\mu}(x,u) = trigausstype2(x, u, [a_1, b_1, c_1, a_2, b_2, c_2, \rho])$is used and we can appreciate the image of the membership function in Fig. 3.1.

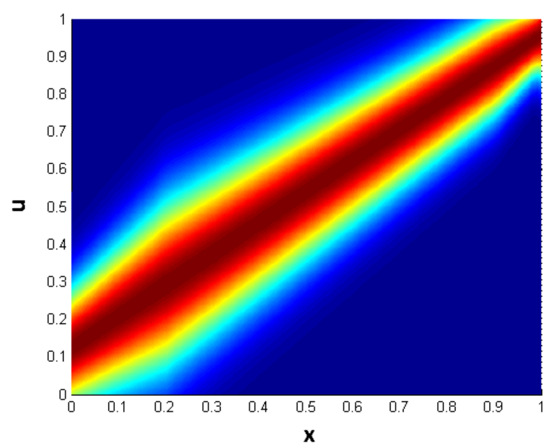


Fig. 3.1. Generalized membership function trigausstype-2

3) After, it is necessary to calculate the α_i cuts for each $M(x_i)$ and $D(x_i)$ using (eq.) to obtain $\mu(M_{iL\alpha_i}(x_i))$, $\mu(M_{iR\alpha_i}(x_i))$ and $\mu(D_{L\alpha_i}(x_i))$, $\mu(D_{R\alpha_i}(x_i))$.

4) Based on (__), the next step is to calculate the λ parameter and the α_i cuts for λ_L and λ_R using the following equations:

$$f\left(\lambda_{L\alpha_i}\right) = \left\{\prod_{i=1}^{n}(1 + M_{iL\alpha_i}(x_i)\,\lambda_{L\alpha_i}\ )\right\} - (1 + \lambda_{L\alpha_i}\ )=0 \quad 3.1$$

$$f\left(\lambda_{R\alpha_i}\right) = \left\{\prod_{i=1}^{n}(1 + M_{iR\alpha_i}(x_i)\,\lambda_{R\alpha_i}\ )\right\} - (1 + \lambda_{R\alpha_i}\ )=0 \quad 3.2$$

once we get $\lambda_{L\alpha_i}$and $\lambda_{R\alpha_i}$ we now have to calculate the fuzzy measures $\mu_{L\alpha_i}(A_i)$ and $\mu_{R\alpha_i}(A_i)$using

$$\mu_{L\alpha_i}(A_1) = \mu_{L\alpha_i}(x_1) \quad 3.3$$

$$\mu_{L\alpha_i}(A_i) = \mu_{L\alpha_i}(x_i) + \mu_{L\alpha_i}(A_{i-1}) + \lambda_{L\alpha_i}\mu_{L\alpha_i}(x_i)\mu_{L\alpha_i}(A_{i-1}) \quad 3.4$$

$$\mu_{R\alpha_i}(A_1) = \mu_{R\alpha_i}(x_1) \quad 3.5$$

$$\mu_{R\alpha_i}(A_i) = \mu_{R\alpha_i}(x_i) + \mu_{R\alpha_i}(A_{i-1}) + \lambda_{R\alpha_i}\mu_{R\alpha_i}(x_i)\mu_{R\alpha_i}(A_{i-1}) \quad 3.6$$

5) Next, we need calculate the Sugeno integral for each $\alpha_i$with

$$\text{h}(\tilde{\sigma}_{\alpha_1}, \tilde{\sigma}_{\alpha_2}, \dots, \tilde{\sigma}_{\alpha_n}) = \sqcup_{I=1}^{n} \left(\sqcap\left[h_{L\alpha_1}, h_{R\alpha_1}\right], \sqcap\left[h_{L\alpha_2}, h_{R\alpha_2}\right], \dots, \sqcap[h_{L\alpha_n}, h_{R\alpha_n}]\right) \quad 3.7$$

where each $\tilde{\sigma}_{\alpha_i}$is an interval of the form

$$\tilde{\sigma}_{\alpha_i} = \sqcup_{i=1}^{n} \left(\sqcap\left(\left[\mu(D_{L\alpha_i}(x_i)), \mu_{L\alpha_i}(A_i)\right], \left[\mu(D_{R\alpha_i}(x_i)), \mu_{R\alpha_i}(A_i)\right]\right)\right) \quad 3.8$$

6) Finally, we calculate the supreme of the $\tilde{\sigma}_{\alpha_i}$with (__)to obtain the GT2SI.

$$h = \sup_{i} \left(h\left(\tilde{\sigma}_{\alpha_i}\right)\right) \quad 3.9$$

To exemplify the proposed method the following parameters are defined below, if we have the information sources given as $\sigma(x_1) = 0.9$, $\sigma(x_2) = 0.6$, $\sigma(x_3) = 0.3$ and the fuzzy densities defined as $\mu c(\boldsymbol{x_1}) = 0.3$, $\mu c(\boldsymbol{x_2}) = 0.4$ and $\mu c(\boldsymbol{x_3}) = 0.1$, associated to each $\sigma(x_i)$, the first step is evaluate each $\sigma(x_i)$ with the membership function trigausstype2(x, u, [-0.5 0.9 1.3 0.2 1.2 1.2 0.1]), to generate the alfa cuts of the fuzzy densities, The representation of the membership function can be found in Fig. 3.2. For each of the α_i cuts we obtain a left interval and right interval of uncertainty.

Using the parameters defined previously, we calculate the α cuts in the points 0.2, 0.4, 0.6, 0.8 and 0.99 for each fuzzy density M(x_i) to obtain $\mu(M_{iL\alpha i}(x_i))$ and $\mu(M_{iR\alpha i}(x_i))$:

$\mu(M_{1L\alpha 0.2}) = 0.1064$, $\mu(M_{1L\alpha 0.4})$ =0.1723, $\mu(M_{1L\alpha 0.6}) = 0.2237$, $\mu(\ M_{1L\alpha 0.8})$ =0.2750, $\mu(M_{1L\alpha 0.99}) = 0.3538$

$\mu(M_{1R\alpha 0.2}) = 0.6436$, $\mu(M_{1R\alpha 0.4})$ =0.5577, $\mu(M_{1R\alpha 0.6}) = 0.5263$, $\mu(\ M_{1R\alpha 0.8})$ =0.4750, $\mu(M_{1R\alpha 0.99}) = 0.3962$

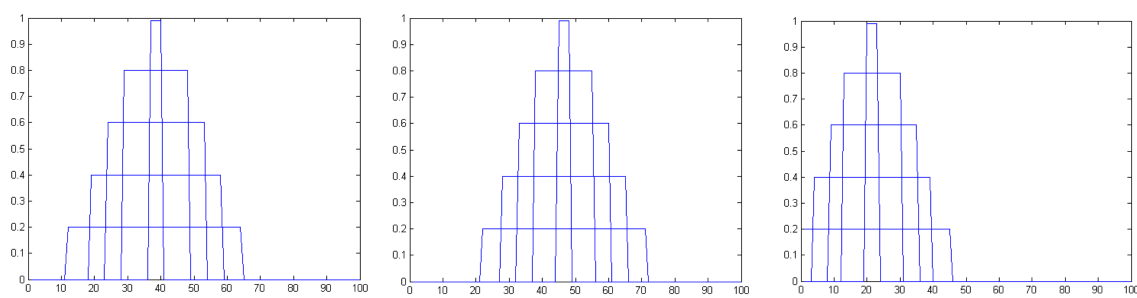
$\mu(M_{2L\alpha 0.2}) = 0.2060$, $\mu(M_{2L\alpha 0.4})$ =0.2680, $\mu(M_{2L\alpha 0.6}) = 0.3162$, $\mu(M_{2L\alpha 0.8})$ =0.3644, $\mu(M_{2L\alpha 0.99} = 0.4384$

$\mu(M_{2R\alpha 0.2}) = 0.7106$, $\mu(M_{2R\alpha 0.4})$ =0.6487, $\mu(\ M_{2R\alpha 0.6}) = 0.6005$, $\mu(M_{2R\alpha 0.8})$ =0.5523, $\mu(M_{2R\alpha 0.99}) = 0.3962$

$\mu(M_{3L\alpha 0.2}) = 0.0100$, $\mu(M_{3L\alpha 0.4})$ =0.0241, $\mu(M_{3L\alpha 0.6}) = 0.0708$, $\mu(M_{3L\alpha 0.8})$ =0.1174, $\mu(M_{3L\alpha 0.99}) = 0.1890$

$\mu(M_{3R\alpha 0.2}) = 0.5425$, $\mu(M_{3R\alpha 0.4})$ =0.3926, $\mu(M_{3R\alpha 0.6}) = 0.3454$, $\mu(M_{3R\alpha 0.8})$ =0.2992, $\mu(M_{3R\alpha 0.99)} = 0.2276$

In the Fig.____ we can appreciate the alpha cuts performed in the parameter of the fuzzy densities assigned to each information source.



a)$\propto(M_1)$=0.3 b) $\propto(M_2)$=0.4 c) $\propto(M_3)$=0.1

Fig 3.2 alpha cuts of the information sources.

On the same way, we calculate the α_i for the information sources D (x_i) to obtain $\mu(D_{L\alpha i}(x_i))$ and $\mu(D_{R\alpha i}(x_i))$.

$\mu(D_{1L\alpha 0.2}) = 0.7041, \mu(D_{1L\alpha 0.4}) = 0.7460, \mu(D_{1L\alpha 0.6}) = 0.7787, \mu(D_{1L\alpha 0.8}) = 0.8114, \mu(D_{1L\alpha 0.99}) = 0.8615$

$\mu(D_{1R\alpha 0.2}) = 1, \mu(D_{1R\alpha 0.4}) = 1, \mu(D_{1R\alpha 0.6}) = 0.9713, \mu(D_{1R\alpha 0.8}) = 0.9386,\ \mu(D_{1R\alpha 0.99}) = 0.8885$

$\mu(D_{2L\alpha 0.2}) = 0.4053, \mu(D_{2L\alpha 0.4}) = 0.4592, \mu(D_{2L\alpha 0.6}) = 0.5012, \mu(D_{2L\alpha 0.8}) = 0.5432, \mu(D_{2L\alpha 0.99}) = 0.6067$

$\mu(D_{2R\alpha 0.2}) = 0.8447, \mu(D_{2R\alpha 0.4}) = 0.7908, \mu(D_{2R\alpha 0.6}) = 0.7488, \mu(D_{2R\alpha 0.8}) = 0.7086, \mu(D_{2R\alpha 0.99}) = 0.6424$

$\mu(D_{3L\alpha 0.2}) = 0.1064, \mu(D_{3L\alpha 0.4}) = 0.1723, \mu(D_{3L\alpha 0.6}) = 0.2237, \mu(D_{3L\alpha 0.8}) = 0.2750, \mu(D_{3L\alpha 0.99}) = 0.3538$

$\mu(D_{3R\alpha 0.2}) = 0.6436, \mu(D_{3R\alpha 0.4}) = 0.5777, \mu(D_{3R\alpha 0.6}) = 0.5263, \mu(D_{3R\alpha 0.8}) = 0.4750, \mu(D_{3R\alpha 0.99}) = 0.3962$

Once the alpha cuts of the fuzzy densities $\mu(M_{iL\alpha i}(x_i))$ and $\mu(M_{iR\alpha i}(x_i))$ are calculated, is necessary estimate the $\lambda_{L\alpha i}$ and $\lambda_{R\alpha i}$, by equations (__).

$\lambda_{L\alpha 0.2} = 22.5836$, $\lambda_{L\alpha 0.4} = 8.1333$, $\lambda_{L\alpha 0.6} = 3.1251$, $\lambda_{L\alpha 0.8} =1.2779$, $\lambda_{L\alpha 0.99} =0.0612$

$\lambda_{R\alpha 0.2} = -0.9162$, $\lambda_{R\alpha 0.4} = -0.8458$, $\lambda_{R\alpha 0.6} = -0.7589$, $\lambda_{R\alpha 0.8} =-0.6274$, $\lambda_{R\alpha 0.99} =-0.2710$

Using (__), the next step is calculate the fuzzy measures $\mu_{L\alpha i}(A_i)$ and $\mu_{R\alpha i}(A_i)$ for each α_i

$\mu_{L\alpha 0.2}(A_1)$=0.1064, $\mu_{L\alpha 0.4}(A_1)$=0.1723, $\mu_{L\alpha 0.6}(A_1)$=0.2237, $\mu_{L\alpha 0.8}(A_1)$=0.2750, $\mu_{L\alpha 0.99}(A_1)$=0.3538

$\mu_{R\alpha 0.2}(A_1)$=0.6436, $\mu_{R\alpha 0.4}(A_1)$=0.5777, $\mu_{R\alpha 0.6}(A_1)$=0.5263, $\mu_{R\alpha 0.8}(A_1)$=0.4750, $\mu_{R\alpha 0.99}(A_1)$=0.3962

$\mu_{L\alpha 0.2}(A_2)$=0.8076, $\mu_{L\alpha 0.4}(A_2)$=0.8159, $\mu_{L\alpha 0.6}(A_2)$=0.7609, $\mu_{L\alpha 0.8}(A_2)$=0.7674, $\mu_{L\alpha 0.99}(A_2)$=0.8017

$\mu_{R\alpha 0.2}(A_2)$=0.9352, $\mu_{R\alpha 0.4}(A_2)$=0.9094, $\mu_{R\alpha 0.6}(A_2)$=0.8869, $\mu_{R\alpha 0.8}(A_2)$=0.8627, $\mu_{R\alpha =0.99}(A_2)$=0.8231

$\mu_{L\alpha 0.2}(A_3)$=1, $\mu_{L\alpha =0.4}(A_3)$=1, $\mu_{L\alpha =0.6}(A_3)$=1, $\mu_{L\alpha =0.8}(A_3)$=1, $\mu_{L\alpha =0.99}(A_3)$=1

$\mu_{R\alpha 0.2}(A_3)$=1, $\mu_{R\alpha 0.4}(A_3)$=1, $\mu_{R\alpha 0.6}(A_3)$=1, $\mu_{R\alpha 0.8}(A_3)$=1, $\mu_{R\alpha 0.99}(A_3)$=1

Next, is necessary perform the calculation of the GT2SI for the α_L cut and the α_R cut in 0.2 in the following way:

$$\sigma_{\alpha 0.2} = \sqcup \Big(\sqcap\,([\mu(D_{L\alpha 0.2}(x_1)), \mu_{L\alpha 0.2}(A_1)], [\mu(D_{R\alpha 0.2}(x_1)), \mu_{R\alpha 0.2}(A_1)]),$$

$$\sqcap\,([\mu(D_{L\alpha 0.2}(x_2)), \mu_{L\alpha 0.2}(A_2)], [\mu(D_{R\alpha 0.2}(x_2)), \mu_{R\alpha 0.2}(A_2)]),$$

$$\sqcap\,([\mu(D_{L\alpha 0.2}(x_3)), \mu_{L\alpha 0.2}(A_3)], [\mu(D_{R\alpha 0.2}(x_3)), \mu_{R\alpha 0.2}(A_3)])\Big)$$

The values obtained above are then replaced to carry out the following calculation:

$$\sigma_{\alpha 0.2} = \sqcup\,(\sqcap\,(([0.7441, 0.1064], [1, 0.6436]),$$

$$\sqcap\,([0.4053, 0.8076], [0.8447, 0.9352]), \sqcap\,([0.1064, 1], [0.6436, 1])))$$

$$\sigma_{\alpha 0.2} = \sqcup\,([0.1064, 0.6436], [0.4053, 0.8447], [0.1064, 0.64336])$$

$$\sigma_{\alpha 0.2} = [0.4053, 0.8447]$$

In the same way, we perform the calculation for the α_L cut and the α_R cut in 0.4:

$$\sigma_{\alpha 0.4} = \sqcup\ (\sqcap\ (([0.7460, 0.1723], [1, 0.5777]),$$

$$\sqcap\ ([0.4592, 0.8159], [0.7908, 0.9094]), \sqcap\ ([0.1723, 1], [0.5777, 1])))$$

$$\sigma_{\alpha 0.4} = \sqcup\ ([0.1723, 0.5777], [0.4592, 0.7908], [0.1723, 0.5777])$$

$$\sigma_{\alpha 0.4} = [0.4592, 0.7908]$$

The result of calculating the $\alpha_L$cut and the $\alpha_R$cut in 0.6 is:

$$\sigma_{\alpha 0.6} = \sqcup\ (\sqcap\ (([0.7787, 0.2237], [0.9713, 0.5263]),$$

$$\sqcap\ ([0.5012, 0.7609], [0.7488, 0.8869]), \sqcap\ ([0.2237, 1], [0.5263, 1])))$$

$$\sigma_{\alpha 0.6} = \sqcup\ ([0.2237, 0.5263], [0.5012, 0.7488], [0.2237, 0.5263])$$

$$\sigma_{\alpha 0.6} = [0.5012, 0.7488]$$

For the α_L cut and the α_R cut in 0.8 the calculation is performed as follows:

$$\sigma_{\alpha 0.8} = \sqcup\ (\sqcap\ (([0.8114, 0.2750], [0.9386, 0.4750]),$$

$$\sqcap\ ([0.5432, 0.7674], [0.7086, 0.8627]), \sqcap\ ([0.2750, 1], [0.4750, 1])))$$

$$\sigma_{\alpha 0.8} = \sqcup\ ([0.2750, 0.4750], [0.5432, 0.7086], [0.2750, 0.4750])$$

$$\sigma_{\alpha 0.8} = [0.5432, 0.7086]$$

In the cut 0.99 the calculation of α_L and α_R is carried out as follows:

$$\sigma_{\alpha 0.99} = \sqcup\ (\sqcap\ (([0.8615, 0.3538], [0.8885, 0.3962]),$$

$$\sqcap\ ([0.6076, 0.8017], [0.6424, 0.8231]), \sqcap\ ([0.3538, 1], [0.3962, 1])))$$

$$\sigma_{\alpha 0.99} = \sqcup\ ([0.3538, 0.3962], [0.6076, 0.6424], [0.3538, 0.3962])$$

$$\sigma_{\alpha 0.99} = [0.6076, 0.6424]$$

Once the SI was calculated for all the α_i cuts, we use (3.9) to obtain an approximation of the GT2SI. Using the data assigned in the previous example, we calculated the SI and the IT2SI. We can find the results obtained for the three methods in Table 3.1. It can be observed that by applying the traditional SI a result of 0.6 was obtained, while using the IT2SI, the uncertainty interval of [0.5, 0.7] was obtained; In this case, (__) was used to calculate an approximation in order to obtain a numerical value representing the output of the IT2SI at the moment of taking it to an actual application. With the GT2SI, an interval of uncertainty was obtained for each of the α cuts, in this case five α cuts were defined, and so we have as output five intervals (one for each α cut).

Table 3.1. Results obtained using SI, IT2SI and GT2SI

Method	**Interval obtained**	**Result**
SI		0.6
IT2SI	[0.5, 0.7]	0.6
GT2SI	[0.4053, 0.8447] [0.4592, 0.7908] [0.5012, 0.7488] [0.5432, 0.7068] [0.6076, 0.6424]	0.625

When working with generalized type-2 fuzzy systems, the challenge is to identify the number of alpha cuts to be performed according to the application, as well as the position of each cut. For this example a sample was obtained by varying both the number of cuts as well as the position of each one. In the Table 3.2 are shows the results obtained. Were conducted 14 tests arbitrarily; It is possible to appreciate that the best result was obtained both with tests 1 and 3 with 5 and 3 alpha cuts respectively, however tests 6 and 8 were also performed using 5 and 3 cuts each, but the results did not were so satisfactory, this is because the positions of the cuts made in each test were different, so it must be taken into account that both the number of cuts and the position play an important role when working with a Generalized type -2 Fuzzy system.

Table 3.2. Results of the variation of number of cuts and the position in a GT2SI.

Test	**Result using GT2SI**	**Number of ∝ *cuts***	**Position of the ∝ cuts**
1	0.625	5	[0.2 0.4 0.6 0.8 0.99]
2	0.6196	6	[0 0.2 0.4 0.6 0.8 1]
3	0.625	3	[0.3 0.6 0.9]
4	0.5929	1	[0]
5	0.6089	2	[0 0.1]
6	0.6143	3	[0 0.1 0.2]
7	0.617	4	[0 0.1 0.2 0.3]
8	0.6186	5	[0 0.1 0.2 0.3 0.4]
9	0.6196	6	[0 0.1 0.2 0.3 0.4 0.5]
10	0.6204	7	[0 0.1 0.2 0.3 0.4 0.5 0.6]
11	0.621	8	[0 0.1 0.2 0.3 0.4 0.5 0.6 0.7]
12	0.6214	9	[0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8]
13	0.6218	10	[0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9]
14	0.6221	11	[0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 0.99]

4. Simulation results

In this chapter simulation results are presented using the proposed method and two study cases to improve or demonstrate the results

4.1.1. Morphological gradient edge detector using GT2SI (MG+GT2SI)

The main objective of this research is to use the GT2SI, as an aggregation method of the Morphological gradient edge detector. Therefore, when applying the morphological gradient edge detection method, the aggregation performed using (2.43) will be replaced by the method proposed in chapter 3.

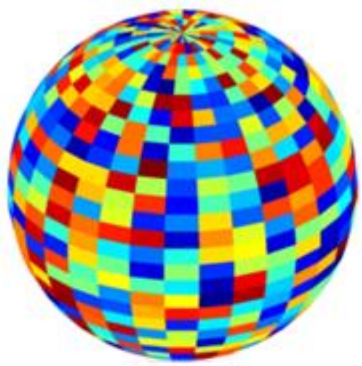


Fig. 4.1 Sphere synthetic image.

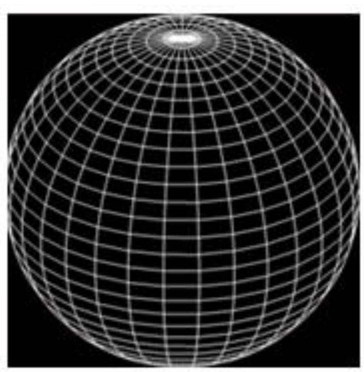
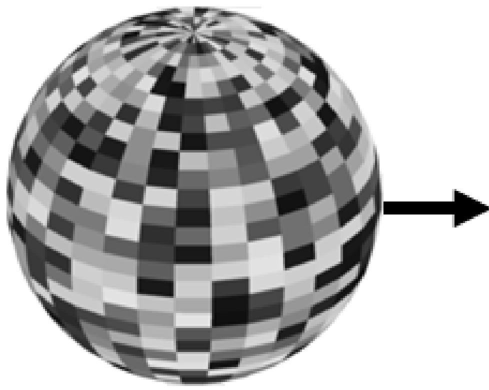


Fig. 4.2 Sphere reference image.



6	**14**	**10**	10	4	3
11	**16**	**17**	20	6	3
11	**20**	**15**	10	5	4
13	16	6	2	2	2
11	16	7	3	4	2
6	4	4	2	3	2

In addition to the visual comparison of the results using real images, we also applied in synthetic images the Pratt's Figure of Merit using (2.44) as performance index to determine the quality of the resulting image.

In this section, the simulation results of the proposed method applied to synthetic images and real images are presented. To carry out the aggregation process we use G_i as the information sources which must be aggregate, in this case instead of using the traditional method of MG (8), we use Choquet integral (CHMG) with equation (13) and the Interval type-2 Choquet Integral (IT2CHMG) using the formulas (23-25) to detect the edges in the images.

The aggregation process of the IT2CHMG is shown in Fig. 4.

Once edge detection with a particular technique is performed, it is necessary to use some evaluation method to determine whether the result is good or better than other edge detection methods existing in digital images.

In the literature, we can find different metrics to evaluate the detected edges, to perform these evaluations, a measure of differences that is frequently used is the quadratic average distance proposed by Pratt [13], and it is also called Figure of Merit (FOM). This measure represents the variation that exists from a real point (calculated) from the ideal edge.

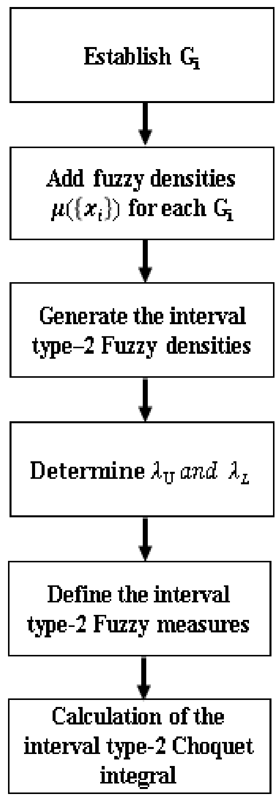


Fig. 4.1. Diagram that represent the integration of gradients.

where I_A represents the number of detected edge points, I_I is the number of points on the ideal edge, d(i) is the distance between the current pixel and its correct position in the reference image and α is a scaling parameter (normally 1/9)**.** To apply (26) we require the synthetic image and its reference. A FOM=1 corresponds to a perfect match between the edge ideal and the detected edge points.

Table 4.2. FOM using GT2SI in the synthetic image of the donut.

Fuzzy density CHMG	**λ**	**FOM**
0.1	6.608	0.8661
0.2	0.7546	0.8712
0.3	-0.4017	0.8736
0.4	-0.7715	0.8732
0.5	-0.9126	0.8771
0.6	-0.9694	0.8785
0.7	-0.9912	0.4824
0.8	-0.9984	0.4611
0.9	-0.9999	0.4387
0.59	-0.9657	0.8787
0.55	-0.9474	0.8777

In Tables 2 and 3, we can observe the parameters used to implement the CHMG method in the synthetic images of the donut and sphere respectively. Each of the four gradients is considered by the aggregation operator as a source of information to which it should be assigned a fuzzy density, which represents the degree of membership or level of importance of that data. In this case, to each of the gradients it was assigned the same fuzzy density.

Various tests in which the FOU parameter was varied from 0.1 to 0.9 were made. After, for each test the calculation of the lambda parameter was performed to carry out the aggregation of the gradients by detecting edges. Finally, was proceeded to the evaluation of the results by using FOM (26).

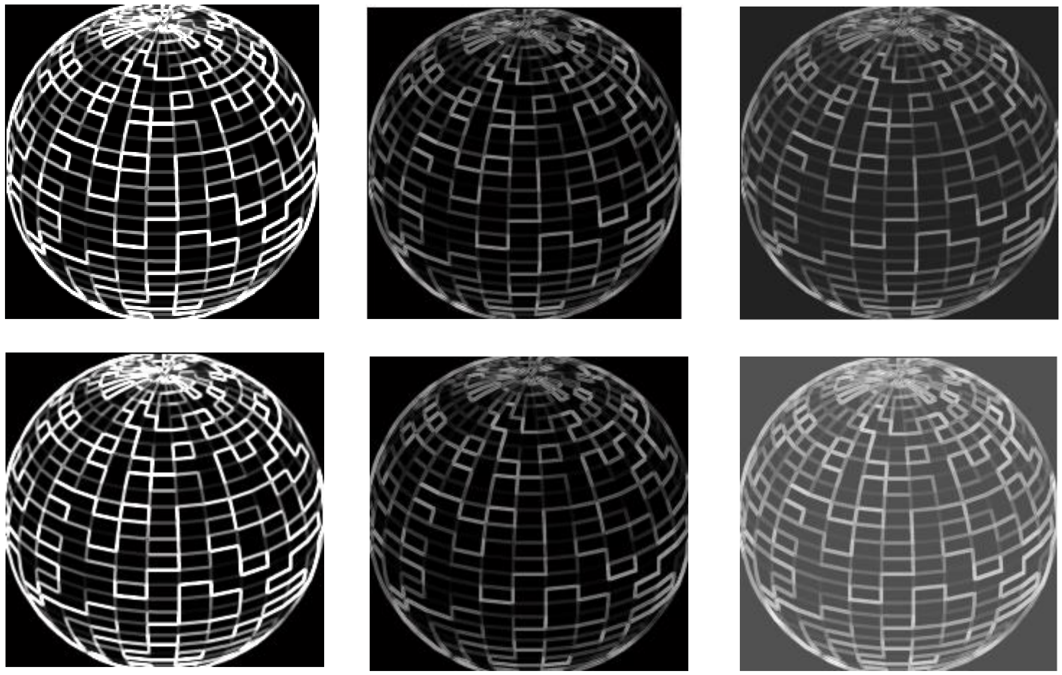
In Table 2 we can find that for the image of the donut, using the CHMG a FOM of 0.8787 was obtained, while for the image of the sphere, a FOM of 0.8236 was achieved; this can be found in Table 3. In both cases the best result was achieved by assigning a fuzzy density of 0.59 to each gradient.

Figures 5 and 6 show the original synthetic images of the donut and the sphere, the reference image is needed to perform the evaluation of the detected edge using the PRATT metric, the image obtained after applying the traditional method of MG using (8) and finally presents the detected edges making variations in the fuzzy densities of each of the information sources from 0.1 to 0.9.

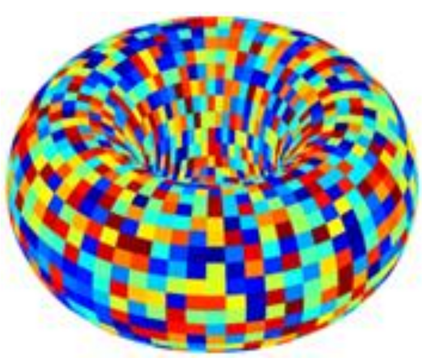
Table 4.3. FOM using CHMG in the synthetic image of the Sphere

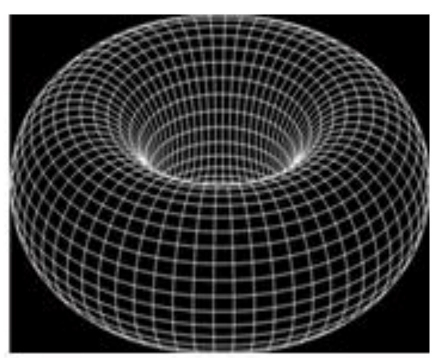
Fuzzy density CHMG	**λ**	**FOM**
0.1	6.608	0.8132
0.2	0.7546	0.8183
0.3	-0.4017	0.8207
0.4	-0.7715	0.8194
0.5	-0.9126	0.8227
0.6	-0.9694	0.8228
0.7	-0.9912	0.4667
0.8	-0.9984	0.4453
0.9	-0.9999	0.4183
0.59	-0.9657	0.8236
0.55	-0.9474	0.8233

MG **SI** **GT2SI**

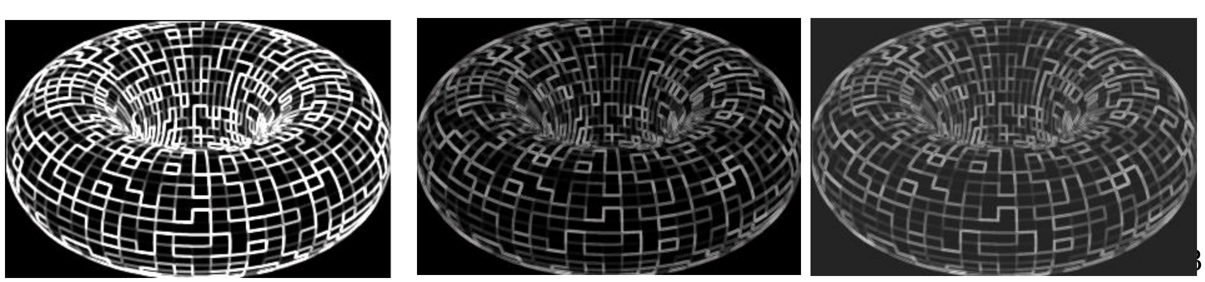


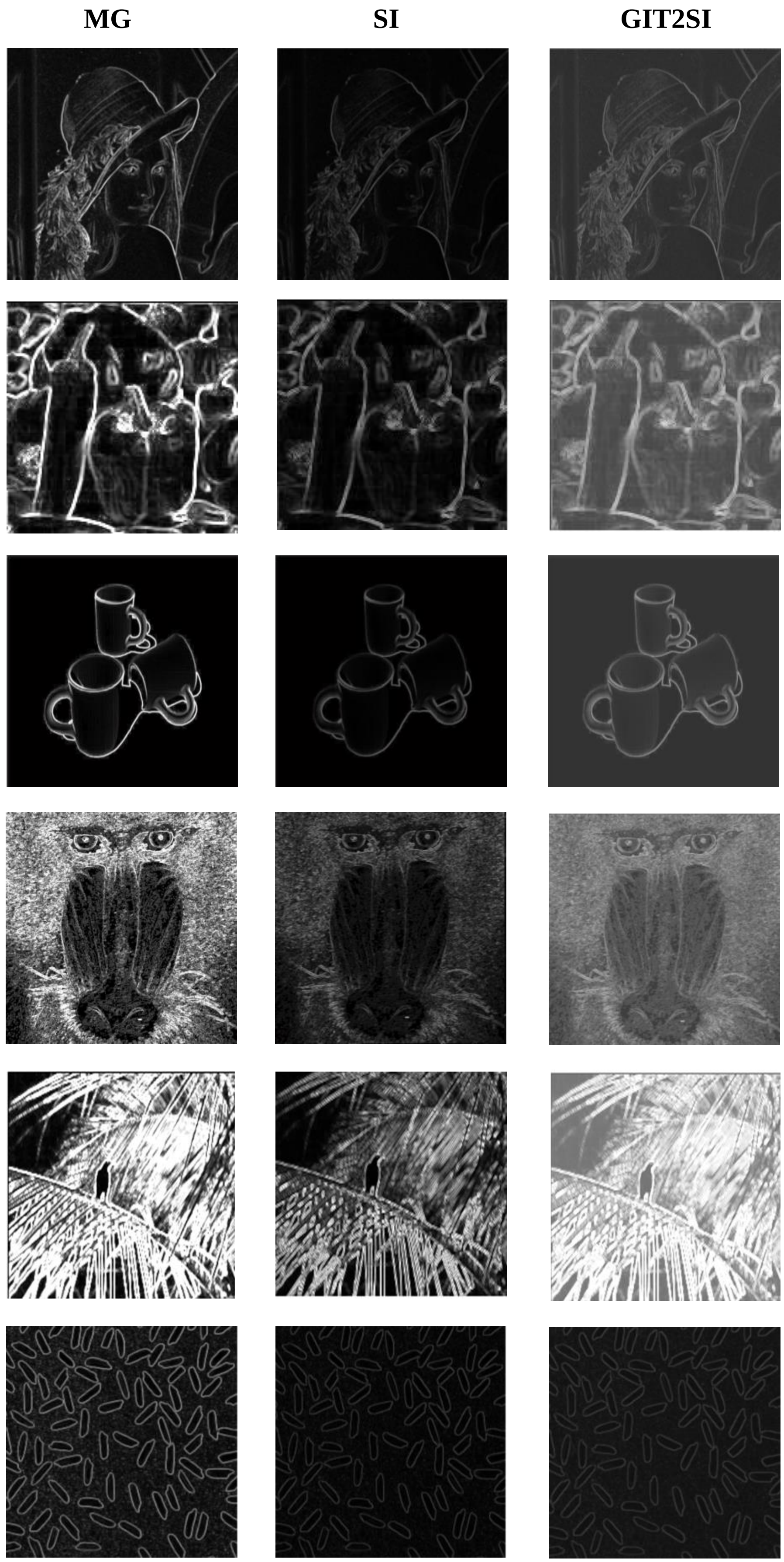
	Sphere			Donut		
	PFOMSGM	PFOMSSI	PFOMSGSI	PFOMSGM	PFOMSSI	PFOMSGSI
mean	0.8744	0.9477	0.9031	0.8199	0.9408	0.8952
max			0.9582			0.9503





MG **SI** **GT2SI**




MG
SI
GIT2SI

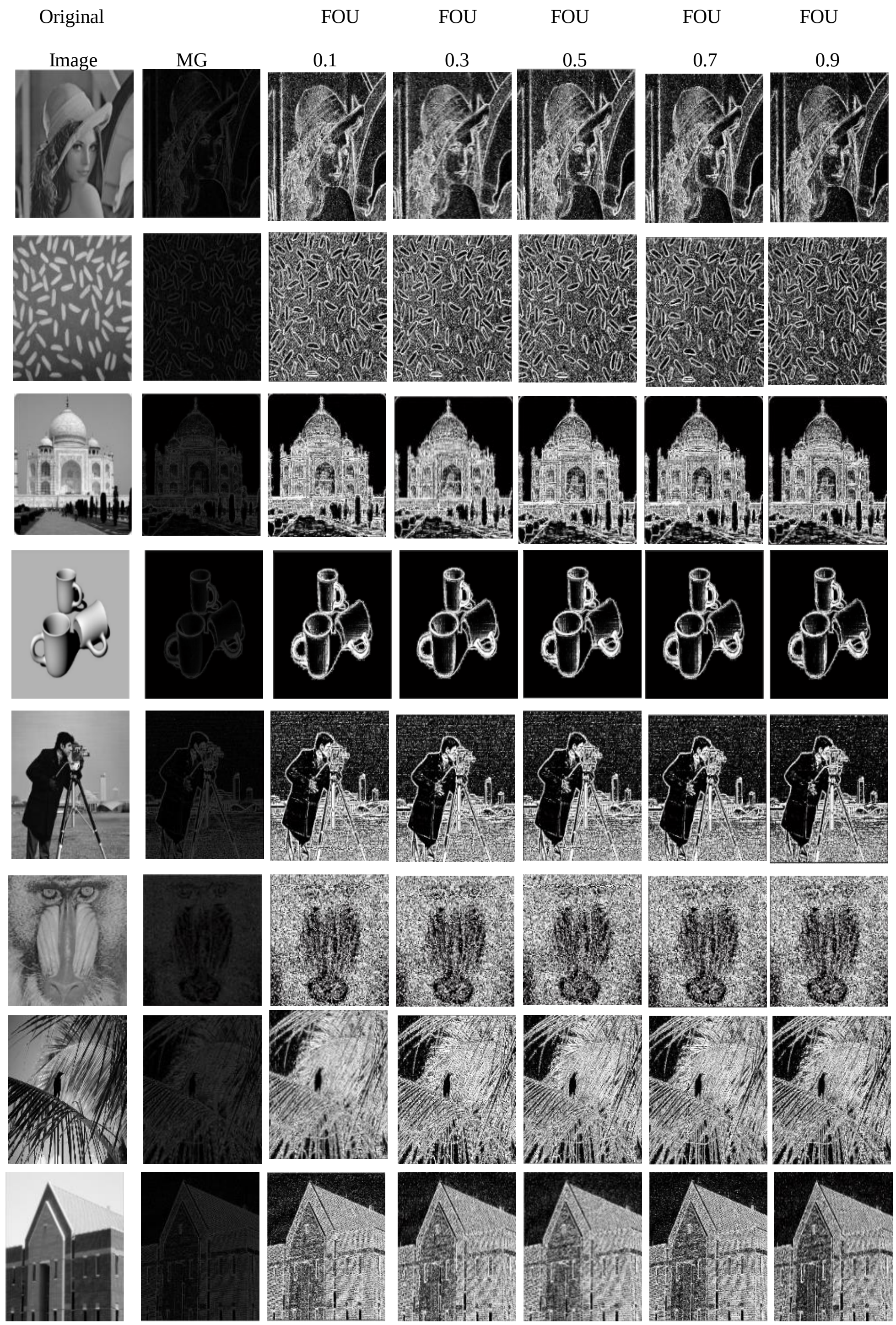


Fig .4.4 Simulation results applying MG and GT2SI in real images.

5. Conclusions and future work

In this thesis, the main objective was to develop a method to extend the fuzzy Sugeno integral using generalized type-2 fuzzy logic. The proposed method was developed and it was demonstrated that works correctly in two very different applications.

In the case of the (MG + GT2SI) presented in Section 4.1, according to the results obtained it was shown that when combining a traditional method of edge detection with the method of aggregation of the Sugeno integral extended with generalized type-2 fuzzy logic, the performance at the time of the edge identification was much better than with traditional methods, so it can be determined that the use of GT2SI is viable to improve image processing methods based on gradient measures.

On the other hand, the experimental results given by the integration of the MNN (MNN+GT2SI) which were presented in Section 4.2 and 4.3, we can conclude that although the aggregation methods SI, IT2SI and GT2SI are able to combine the information satisfactorily and produce the correct result, the combination of the SI with generalized type-2 fuzzy system is able to handle a higher level of uncertainty that IT2SI and SI, what you can see In the results shown in the tables __, for the cases of faces recognition using the ORL and Cropped Yale databases.

With the development of this thesis work it can be determined that when we have a system or problem in which there is a high level of uncertainty, it is possible to make use of generalized type-2 fuzzy systems, since, due to their nature, they may have greater uncertainty management, especially compared to fuzzy systems and fuzzy type-2 systems. The results obtained demonstrate that there are considerable advantages when using them, however, for the moment the proposed method of GT2SI cannot be applied in real-time systems due to the processing power that is required to be able to carry out that homework.

As future work, it is proposed to use optimization techniques in order to explore and find the appropriate parameters for the generalized type-2 membership function, so that the performance of the proposed system is optimum, as well as perform an exploration to determine if the number of alpha cuts used in pattern recognition and edge detection applications was the one that allows to take the most advantage of the aggregation operator.

It is also intended to make use of the operator in other application problems where it is necessary to perform numerical aggregation, as well as in problems where there is an amount n of diverse sources of information. In addition, it is proposed to create a system by means of which it is possible to analyze the data contained in each source of information so that, based on them, the diffuse densities assigned to each source can be automatically generated. It is also suggested to use the operator in other application problems where it is necessary to perform numerical aggregation, as well as in problems where there is an amount n of diverse sources of information.

This thesis document presents an approximation of the GT2SI, however it is still necessary to analyze if it is possible to carry out the calculations in some other way.

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Annexed

Pseudocode of the Sugeno integral.

INPUT: Number of information sources n; information sources x_1, x_2,…, x_n; fuzzy densities of information sources M_1, M_2,…,$M_n \in (0,1)$.

OUTPUT: Sugeno integral h(σ(x_1), σ(x_2),…..,σ(x_n)).

Step 1: calculate λ finding the root of the equation
$f(\lambda) = \{\prod_{i=1}^{n}(1 + M_i(x_i)\lambda)\} - (1 + \lambda)$=0

Step 2: Fuzzify variable x_i.

$$D_i = \{x_, \mu_{Di}(x) | x \epsilon X\},\ \mu D_i(x) \in [0, 1]$$

Step 3: Reorder M_i with respect to D (x_i) in descending order

Step 4: Calculate fuzzy measures for each data with (8) (9).

Step 5: Calculate Sugeno integral with (19) or (20).

Step 6: OUTPUT Sugeno integral.

Stop

Pseudocode of the Generalized Type-2 Sugeno integral.

Input: Number of information sources *n*, information sources D(x_i), Fuzzy densities of the information sources M(x_i) ∈ (0,1).

Output: Generalized type-2 Sugeno integral *h*.

Step 1: Evaluate each D(x_i) and M(x_i) with the function

$$\tilde{\mu}(x, u) = trigausstype2(x, u, [a_1, b_1, c_1, a_2, b_2, c_2, \rho])$$

Step 2: Calculate the α_i cuts for each M(x_i) and D(x_i)

$\mu(M_{iL\alpha_i}(x_i))$, $\mu(M_{iR\alpha_i}(x_i))$

$\mu(D_{L\alpha_i}(x_i))$, $\mu(D_{R\alpha_i}(x_i))$

Step 3: Calculate λ and the α_i cuts for λ_L and λ_R using

$$f(\lambda_{L\alpha_i}) = \left\{\prod_{i=1}^{n}(1 + M_{iL\alpha_i}(x_i)\ \lambda_{L\alpha_i}\ )\right\} - (1 + \lambda_{L\alpha_i}\ )=0$$

$$f(\lambda_{R\alpha_i}) = \left\{\prod_{i=1}^{n}(1 + M_{iR\alpha_i}(x_i)\ \lambda_{R\alpha_i}\ )\right\} - (1 + \lambda_{R\alpha_i}\ )=0$$

Step 4: Calculate the Fuzzy measures $\mu_{L\alpha_i}(A_i)$ and $\mu_{R\alpha_i}(A_i)$

$$\mu_{L\alpha_i}(A_1) = \mu_{L\alpha_i}(x_1)$$

$$\mu_{L\alpha_i}(A_i) = \mu_{L\alpha_i}(x_i) + \mu_{L\alpha_i}(A_{i-1}) + \lambda_{L\alpha_i}\mu_{L\alpha_i}(x_i)\mu_{L\alpha_i}(A_{i-1})$$

$$\mu_{R\alpha_i}(A_1) = \mu_{R\alpha_i}(x_1)$$

$$\mu_{R\alpha_i}(A_i) = \mu_{R\alpha_i}(x_i) + \mu_{R\alpha_i}(A_{i-1}) + \lambda_{R\alpha_i}\mu_{R\alpha_i}(x_i)\mu_{R\alpha_i}(A_{i-1})$$

Step 5: For each α_i calculate the Sugeno integral $h(\tilde{\sigma}_{\alpha_i})$

$$\text{h}(\tilde{\sigma}_{\alpha_1}, \tilde{\sigma}_{\alpha_2}, \dots, \tilde{\sigma}_{\alpha_n}) = \sqcup_{I=1}^{n} \left(\prod\left[h_{L\alpha_1}, h_{R\alpha_1}\right], \prod\left[h_{L\alpha_2}, h_{R\alpha_2}\right], \dots, \prod[h_{L\alpha_n}, h_{R\alpha_n}]\right)$$

where

$$\tilde{\sigma}_{\alpha_i} = \sqcup_{i=1}^{n} \left(\sqcap \left(\left[\mu(D_{L\alpha_i}(x_i)), \mu_{L\alpha_i}(A_i)\right], \left[\mu(D_{R\alpha_i}(x_i)), \mu_{R\alpha_i}(A_i)\right]\right)\right)$$

Step 6: Calculate the GT2SI $h = \sup_{i} \left(h(\tilde{\sigma}_{\alpha_i})\right)$